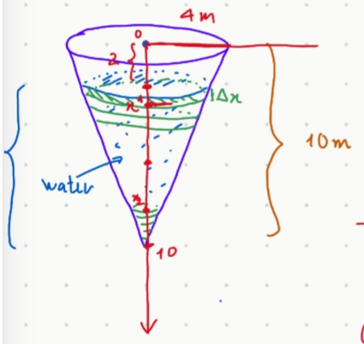
June 6th, 2024

Lecture 11

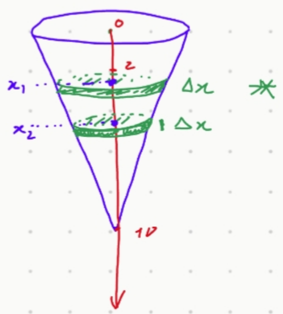
§6.4. Work (continue).

Example: A tank has the shape of an inverted circular cone with height 10m, base radius 4m.
It is filled with water to the height of 8 m.
Find the work required to empty the tank by pumping all of the water to the top of the tank.
(The density of water is 1000 kg / m³).



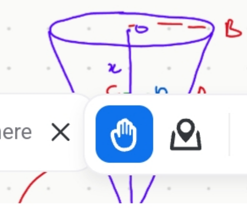
We divide the water into smaller layers by dividing the interval $[2,10]$ into n subintervals of length $\Delta x = \frac{10-2}{n}$ (m)

Take one layer:
↓
Compute the work to pump that layer to the top of the tank
↓
Sum over all layers
↓
Get the work.



1) Compute the work to pump one layer
$W = F \cdot d$ F: Force, d: distance (height).
$F = m \cdot g$ m: mass, $g = 9.8 \ m/s^2$
m = density × volume

i) Mass of one layer:
mass = density × volume



2) Compute

The layer has
The base is a disk of radius r

= 156

2) Compute the force

$F = m \cdot g$

$= 160\pi(10-x)^2 \Delta x \cdot 9.8$

$= 1568\pi(10-x)^2 \Delta x.$

3) Compute the work to pump 1 layer

$W = F \cdot d$

$= (1568\pi(10-x)^2 \Delta x) \cdot x$

$= 1000 \cdot 4\pi \,(10-x)\, \Delta x.$

2) Sum all the works

$$W_{all} = \sum_{i=1}^{n} W_i$$ where W_i is the work done to raise layer x_i

$$= \sum_{i=1}^{n} 1568\pi\, x_i\,(10-x_i)^2\, \Delta x$$

$$= \int_2^{10} 1568\pi x\,(10-x)^2\, dx$$

$$= 1568\pi \int_2^{10} x\,(10-x)^2\, dx$$

$$= 1568\pi \int_2^{10} x\,(100-20x+x^2)\, dx$$

$$= 1568\pi \int_2^{10} (100x - 20x^2 + x^3)\, dx$$

$$= 1568\pi \left(50x^2 - \frac{20x^3}{3} + \frac{x^4}{4}\right)\Big|_2^{10}$$

$$= 1568\pi\, \frac{2048}{3}$$

$$\approx 3.4 \times 10^6\ J.$$

§ 6.5. Average value of a function.

Let f is a continuous function on $[a,b]$

Theorem: Average value of f on the interval $[a,b]$ is

$$f_{a\ldots} = \frac{1}{\ldots}\int_a^b f(x)\,dx$$

Ex: Find the average value of $f(x) = 1 + x^2$ on $[-1, 2]$

§6.5. Average value of a function.

Let f is a continuous function on $[a,b]$

Theorem: Average value of f on the interval $[a,b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex: Find the average value of $f(x) = 1+x^2$ on $[-1, 2]$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx$$

$$= \frac{1}{3}\left(x + \frac{x^3}{3}\right)\Big|_{-1}^{2}$$

$$= \frac{1}{3}\left(\left(2 + \frac{2^3}{3}\right) - \left((-1) + \frac{(-1)^3}{3}\right)\right)$$

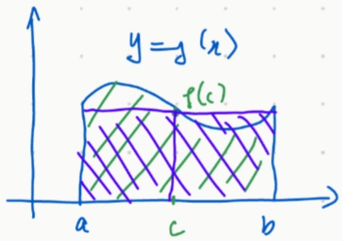
$$= 2.$$

The Mean Value Theorem for integrals

If f is a continuous function on $[a,b]$, then there is $c \in [a,b]$ such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = f(c)(b-a)$$



Area under the graph $y = f(x)$, $x = a$, $x=b$

$\approx f(c)(b-a)$

Example: f ...

MVT $\Rightarrow$ there ...

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$\approx f(c)(b-a)$

Example: $f(x) = 1+x^2$ continuous on $[-1, 2]$.
MVT $\Rightarrow$ there is c in $[-1, 2]$ such that

$$f(c) = f_{avg} = 2.$$

$$\Rightarrow \quad 1 + c^2 = 2$$

$$\Rightarrow \quad c^2 = 1$$

$$\Rightarrow \quad \left[\begin{array}{l} c = 1 \\ c = -1 \end{array}\right.$$

Since $1, -1 \in [-1, 2] \Rightarrow$ both $-1, 1$ satisfy MVT.

Example Find average value of $f = \sqrt{x}$ on $[0,4]$
Find c satisfies MVT.

$$f_{avg} = \frac{1}{4}\int_0^4 \sqrt{x}\, dx = \frac{1}{4} \left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^4 = \frac{1}{2} 4^{1/2} = 1$$

§ 7.1 Integration by parts.

Product rule

$$\left(f(x)\, g(x)\right)' = f(x)\, g'(x) + f'(x)\, g(x)$$

$$\Rightarrow \quad f(x)\, g'(x) = \left(f(x)\, g(x)\right)' - f'(x)\, g(x).$$

Taking antiderivative both sides.

$$\int f(x) g'(x)\, dx = \int \left(f(x) g(x)\right)' dx - \int f'(x) g(x)\, dx$$

$$\Rightarrow \quad \boxed{\int f(x) g'(x)\, dx = f(x) g(x) - \int f'(x) g(x)\, dx}$$

Formula

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$$\int f(x)g'(x)\,dx = \int \big(f(x)g(x)\big)'\,dx - \int f'(x)g(x)\,dx$$

$$\Rightarrow \boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

↳ Formula for integration by parts.

$$\int u\,dv = uv - \int v\,du$$

$f = u$, $g = v$

Example. $\int x \sin x\,dx$

We can not $\begin{cases} \text{use antiderivative table.} \\ \text{use substitution.} \end{cases}$

Choose $\begin{cases} u = x \\ dv = \sin x\,dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = -\cos x \end{cases}$

Apply formula for integration by parts

$$\int x\sin x\,dx = \int u\,dv = uv - \int v\,du$$

$$= x(-\cos x) - \int(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C.$$

Check: $(-x\cos x + \sin x + C)' = x\sin x$

Example. Evaluate $\int \ln x\,dx.$

$\Rightarrow$ choose it in a way that after taking derivative the function is simple

$\begin{cases} u = \ln x \\ dv = dx \end{cases}$ $\begin{cases} du = \frac{1}{x}\,dx \end{cases}$

$\ldots sx + \sin x + C$

Check: $(-x\cos x + \sin x + C)' = x\sin x$

Example: Evaluate $\int \ln x \, dx$.

→ choose it in a way that after taking derivative the function is simple

Choose $\begin{cases} u = \ln x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x} dx \\ v = x \end{cases}$

Integration by parts

$$\int \ln x \, dx = uv - \int v \, du = x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C.$$

Ex. Find $\int t^2 e^t \, dt$.

$\begin{cases} u = t^2 \\ dv = e^t dt \end{cases} \Rightarrow \begin{cases} du = 2t \, dt \\ v = e^t \end{cases}$

Thus $\int t^2 e^t \, dt = uv - \int v \, du = t^2 e^t - \int e^t (2t) \, dt$

$$= t^2 e^t - 2\int t e^t \, dt \qquad (1)$$

We need to compute $\int t e^t \, dt$

Choose $\begin{cases} u = t \\ dv = e^t dt \end{cases} \Rightarrow \begin{cases} du = dt \\ v = e^t \end{cases}$

Thus $\int t e^t \, dt = uv - \int v \, du$

$$= t e^t - \int e^t \, dt$$

$$= t e^t - e^t + C. \qquad (2)$$

Combine (1) and (2) $\Rightarrow \int t^2 e^t \, dt = t^2 e^t - 2(t e^t - e^t) + C.'$

Example: $\int e^x \sin x \, dx$

$u = e^x$

$du = \sin x \, dx \Rightarrow \quad v = -\cos x$

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$\ldots e^{2t}\,dt = te^{2t} - \ldots$

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Example: $\int e^x \sin x \, dx$

$$\begin{cases} u = e^x \\ dv = \sin x \, dx \end{cases} \Rightarrow \begin{cases} du = e^x \, dx \\ v = -\cos x \end{cases}$$

$$I = \int e^x \sin x \, dx = e^x(-\cos x) - \int (-\cos x) e^x \, dx$$

$$= -e^x \cos x + \int \cos x \, e^x \, dx \quad (1)$$

Compute the rest $\text{II} = \int \cos x \, e^x \, dx$

$$\begin{cases} u = e^x \\ dv = \cos x \, dx \end{cases} \Rightarrow \begin{cases} du = e^x \\ v = \sin x \end{cases}$$

$$\text{II} = \int \cos x \, e^x \, dx = \sin x \, e^x - \underline{\int e^x \sin x \, dx} \quad (2)$$

$$I = \int e^x \sin x = -e^x \cos x + \sin x \, e^x - \int e^x \sin dx$$

$$\Rightarrow I = -e^x \cos x + e^x \sin x - I$$

$$\Rightarrow 2I = e^x(\sin x - \cos x) + C$$

$$\Rightarrow I = \frac{e^x(\sin x - \cos x)}{2} + C$$

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